



The Open
University

MST125

Essential mathematics 2

Exercise Booklet 7

1 Revision of integration methods

Exercise 1

Find each of the following integrals by first modifying the integrand and then using the table of standard indefinite integrals.

$$(a) \int \frac{1}{e^{3-2x}} dx \quad (b) \int \sqrt{x}(x+1)^2 dx$$

$$(c) \int \frac{7}{9x^2+1} dx \quad (d) \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$(e) \int x^3 \left(\frac{1}{x^2} + \frac{1}{x} \right)^2 dx$$

$$(f) \int \frac{1}{\sin x \tan x} dx$$

Exercise 2

Use integration by substitution to find the following integrals.

$$(a) \int 3x^2 e^{x^3} dx \quad (b) \int \frac{x^2}{3x^3+5} dx$$

$$(c) \int \frac{3x}{\sqrt{x-2}} dx \quad (d) \int \frac{(\ln x)^2}{x} dx$$

$$(e) \int x\sqrt{2x-3} dx$$

$$(f) \int x \cos(x^2-4) dx \quad (g) \int \frac{1}{e^{2x}+1} dx$$

(Hint for part (g): start by writing the integrand in the form 'constant + algebraic fraction'. To do this, first write the numerator as $e^{2x} + 1 - e^{2x}$.)

Exercise 3

Use integration by substitution to evaluate the following definite integrals.

$$(a) \int_0^{\pi/2} \frac{\cos x}{4 + \sin x} dx$$

$$(b) \int_{-2}^{-1} (2x+1)(x+2)^3 dx$$

$$(c) \int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}} dx$$

Exercise 4

Use integration by parts to find the following integrals.

$$(a) \int 2x \sec^2 x dx \quad (b) \int x^2 e^{x/2} dx$$

$$(c) \int (x-1)(x+2)^4 dx$$

$$(d) \int x \ln x dx \quad (e) \int x^2 (\ln x)^2 dx$$

2 Partial fractions of proper rational expressions

Exercise 5

Find the partial fraction expansions of the following rational expressions.

$$(a) \frac{4x-1}{(x+1)(x-4)}$$

$$(b) \frac{x+5}{3(1-2x)(1+x)}$$

$$(c) \frac{4x+8}{x(3x+2)(x-2)}$$

$$(d) \frac{5x^2+3x+4}{(x-1)(2x+1)(x+2)}$$

Exercise 6

Find the partial fraction expansions of the following rational expressions with repeated linear factors.

$$(a) \frac{3x+2}{(x-1)(x+2)^2} \quad (b) \frac{x+1}{(x+3)^2}$$

$$(c) \frac{2x^2+3}{(x+1)^3}$$

Exercise 7

Find the partial fraction expansions of the following rational expressions, each of which has a quadratic factor in the denominator.

$$(a) \frac{x+3}{(x-1)(x^2-3x+4)}$$

$$(b) \frac{2}{(x-3)(x^2+2)}$$

$$(c) \frac{x^2+x+1}{x(2x^2+3x+2)}$$

Exercise 8

Use partial fractions to find the following integrals.

$$(a) \int \frac{2x-7}{x^2-x-2} dx$$

$$(b) \int \frac{x^2-3x-9}{x^3-6x^2+9x} dx$$

$$(c) \int \frac{6x^2-x+30}{2(2x+3)(x^2+9)} dx$$

3 Partial fractions of improper rational expressions

Exercise 9

Use polynomial long division to write each of the following improper rational expressions as the sum of a polynomial expression and a proper rational expression.

$$(a) \frac{8x^3-6x^2+x+5}{2x+1}$$

$$(b) \frac{2x^3+3x^2-3x+1}{x^2+1}$$

$$(c) \frac{8x^4+15x-30}{2x^2-x+4}$$

Exercise 10

Find the partial fraction expansions of the following improper rational expressions.

$$(a) \frac{x^2-4x+1}{x(x+1)} \quad (b) \frac{x^3+3x^2-3x-11}{x^2+3x-4}$$

Exercise 11

Find the following integrals of improper rational expressions.

$$(a) \int \frac{x^2+1}{x(x+3)} dx \quad (b) \int \frac{x^4}{x^2-2x+1} dx$$

4 Graphs of rational functions

Exercise 12

Find the domains of the following rational functions. Express your answers using interval notation.

$$(a) k(x) = \frac{x^2+7}{x+3} \quad (b) f(x) = \frac{3}{(x+2)^2}$$

$$(c) g(x) = \frac{x}{x^2-3x-4}$$

$$(d) h(x) = \frac{3}{2x^2+3x}$$

Exercise 13

Find the intercepts of the following rational functions.

$$(a) f(x) = \frac{3}{(x+2)^2}$$

$$(b) g(x) = \frac{x}{x^2-3x-4}$$

$$(c) h(x) = \frac{3}{2x^2+3x} \quad (d) k(x) = \frac{x^2+7}{x+3}$$

$$(e) l(x) = \frac{2x-5}{3-x}$$

Exercise 14

Consider the rational function

$$k(x) = \frac{x^2 + 7}{x + 3}.$$

- Find the coordinates of the stationary points of k .
- Determine the intervals on which k is increasing or decreasing.
- Determine whether each stationary point of k is a local maximum, a local minimum or a horizontal point of inflection.

Exercise 15

For each of the following rational functions, write down the equations of its vertical asymptotes and determine the behaviour of the function near these asymptotes.

- $p(x) = \frac{3}{2x^2}$
- $k(x) = \frac{x^2 + 7}{x + 3}$
- $g(x) = \frac{x}{x^2 - 3x - 4}$

Note: for part (b) you can use the solution to Exercise 14(b).

Exercise 16

For each of the following functions, determine its asymptotic behaviour as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, and find the equations of any horizontal or slant asymptotes.

- $f(x) = \frac{1 - 2x^2}{1 + x^2}$
- $f(x) = \frac{4x^3 - 1}{2x^2 + 5}$
- $f(x) = \frac{8x^4 + 16x^3 + 7x - 11}{8x^3 + 27}$

Exercise 17

State whether the following functions are odd, even or neither. Give a reason in each case.

- $f(x) = \frac{1}{2}x^2 + x$
- $f(x) = \frac{2}{x} + x$
- $f(x) = \frac{2x^4 - x^2 + 1}{x^2 + 5}$

Exercise 18

Sketch the graphs of the following rational functions, by following the strategy 'To sketch the graph of a rational function' that is given in Unit 7 and also in the *Handbook*.

- $f(x) = \frac{5}{(x - 3)^2}$
- $f(x) = -\frac{x^3}{x^2 + 3}$
- $f(x) = \frac{1}{x^2 + 4x + 3}$
- $f(x) = \frac{4x}{2x^2 + 3}$

5 More integration methods

Exercise 19

Use suitable trigonometric identities to find the following integrals.

- $\int \cos(2x) \cos(3x) \, dx$
- $\int \sin^5 x \cos^4 x \, dx$
- $\int \frac{\cot x}{1 + \cot^2 x} \, dx$

Exercise 20

Use suitable trigonometric substitutions to find the following integrals.

- $\int \frac{\sqrt{x^2 - 9}}{2x} \, dx \quad (x \geq 3)$
- $\int \frac{1}{x^2 \sqrt{x^2 + 1}} \, dx$

Exercise 21

Find each of the following integrals by rearranging the integrand and then using a suitable substitution.

(Hint: in all three cases it may help to complete the square.)

(a) $\int \frac{1}{x^2 - 4x + 5} dx$

(b) $\int \frac{x + 3}{\sqrt{2x - x^2}} dx$

(c) $\int \frac{x + 3}{\sqrt{x^2 + 6x + 5}} dx \quad (x \geq -1)$

6 Hyperbolic functions

Exercise 22

Find the following values of $\sinh x$, $\cosh x$ and $\tanh x$. A calculator is not needed for parts (a) and (b). In parts (c) and (d), give your answers to four decimal places.

(a) $\sinh(\ln 2)$ (b) $\tanh(\ln 3)$

(c) $\cosh(-5)$ (d) $\sinh^{-1} 2$

Exercise 23

Prove the following identities.

(a) $\sinh(2x) = 2 \sinh x \cosh x$

(b) $\coth^2 x - 1 = \operatorname{cosech}^2 x$

(c) $\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x}$

Exercise 24

Differentiate the following functions.

(a) $f(x) = \tanh(2x)$

(b) $f(x) = \sinh x \tanh x$

(c) $f(x) = e^{\cosh x}$

Exercise 25

Find the following integrals.

(a) $\int \cosh(2x + 3) dx$

(b) $\int \cosh^3 x dx$

(c) $\int x \sinh(2x) dx$

Exercise 26

Use suitable hyperbolic substitutions to find the following integrals.

(a) $\int \frac{1}{\sqrt{9 + x^2}} dx$

(b) $\int \frac{1}{\sqrt{4x^2 - 1}} dx \quad (x \geq \frac{1}{2})$

(Hint: write $\sqrt{4x^2 - 1}$ as $\sqrt{4(x^2 - \frac{1}{4})}$.)

(c) $\int \frac{x + 1}{\sqrt{x^2 + 1}} dx$

7 Choosing integration methods

Exercise 27

Find the following integrals.

(a) $\int (3x + 2)^3 dx$ (b) $\int \frac{1}{\sqrt{3x + 2}} dx$

(c) $\int 4xe^{-x^2} dx$ (d) $\int \sqrt{9 + x^2} dx$

(e) $\int \frac{x^2}{\sqrt{4 - x^2}} dx$ (f) $\int \frac{1}{x^2 + 7x + 12} dx$

(g) $\int \frac{1}{2x^2 + 6} dx$ (h) $\int \frac{x^3 + 4}{x^2 + 2x + 1} dx$

(i) $\int (x - 4) \sin(3x) dx$

(j) $\int 4 \sin(2x) \cos(2x) dx$

(k) $\int \frac{3x}{4x^2 + 1} dx$

Solutions to exercises

Solution to Exercise 1

$$\begin{aligned}
 \text{(a)} \quad \int \frac{1}{e^{3-2x}} dx &= \int e^{-(3-2x)} dx \\
 &= \int e^{2x-3} dx \\
 &= \int e^{2x} e^{-3} dx \\
 &= \frac{1}{e^3} \int e^{2x} dx \\
 &= \frac{1}{e^3} \times \frac{1}{2} e^{2x} + c \\
 &= \frac{1}{2} e^{2x-3} + c.
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int \sqrt{x} (x+1)^2 dx &= \int x^{1/2} (x^2 + 2x + 1) dx \\
 &= \int (x^{5/2} + 2x^{3/2} + x^{1/2}) dx \\
 &= \frac{2}{7} x^{7/2} + \frac{4}{5} x^{5/2} + \frac{2}{3} x^{3/2} + c.
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{7}{9x^2+1} dx &= 7 \int \frac{1}{1+(3x)^2} dx \\
 &= 7 \times \frac{1}{3} \tan^{-1}(3x) + c \\
 &= \frac{7}{3} \tan^{-1}(3x) + c.
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \frac{\sin^2 x}{\cos^2 x} dx &= \int \tan^2 x dx \\
 &= \int (\sec^2 x - 1) dx \\
 &= \tan x - x + c.
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int x^3 \left(\frac{1}{x^2} + \frac{1}{x} \right)^2 dx &= \int x^3 \left(\frac{1}{x^4} + \frac{2}{x^3} + \frac{1}{x^2} \right) dx \\
 &= \int \left(\frac{1}{x} + 2 + x \right) dx \\
 &= \ln |x| + 2x + \frac{1}{2} x^2 + c.
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad \int \frac{1}{\sin x \tan x} dx &= \int \frac{1}{\sin x} \times \frac{1}{\tan x} dx \\
 &= \int \operatorname{cosec} x \cot x dx \\
 &= -\operatorname{cosec} x + c.
 \end{aligned}$$

Solution to Exercise 2

(a) Let $u = x^3$; then $\frac{du}{dx} = 3x^2$. So

$$\begin{aligned}
 \int 3x^2 e^{x^3} dx &= \int e^{x^3} (3x^2) dx \\
 &= \int e^u du \\
 &= e^u + c \\
 &= e^{x^3} + c.
 \end{aligned}$$

(b) Let $u = 3x^3 + 5$; then $\frac{du}{dx} = 9x^2$. So

$$\begin{aligned}
 \int \frac{x^2}{3x^3+5} dx &= \frac{1}{9} \int \frac{1}{3x^3+5} (9x^2) dx \\
 &= \frac{1}{9} \int \frac{1}{u} du \\
 &= \frac{1}{9} \ln |u| + c \\
 &= \frac{1}{9} \ln |3x^3+5| + c.
 \end{aligned}$$

(c) Let $u = x - 2$; then $\frac{du}{dx} = 1$. Also,

$x = u + 2$. So

$$\begin{aligned}
 \int \frac{3x}{\sqrt{x-2}} dx &= 3 \int \frac{u+2}{\sqrt{u}} du \\
 &= 3 \int (u^{1/2} + 2u^{-1/2}) du \\
 &= 2u^{3/2} + 12u^{1/2} + c \\
 &= 2\sqrt{u}(u+6) + c \\
 &= 2\sqrt{x-2}(x+4) + c.
 \end{aligned}$$

(d) Let $u = \ln x$; then $\frac{du}{dx} = \frac{1}{x}$. So

$$\begin{aligned}
 \int \frac{(\ln x)^2}{x} dx &= \int (\ln x)^2 \left(\frac{1}{x} \right) dx \\
 &= \int u^2 du \\
 &= \frac{1}{3} u^3 + c \\
 &= \frac{1}{3} (\ln x)^3 + c.
 \end{aligned}$$

- (e) Let $u = 2x - 3$; then $\frac{du}{dx} = 2$. Also,

$$x = \frac{1}{2}(u + 3). \text{ So}$$

$$\begin{aligned} & \int x\sqrt{2x-3} \, dx \\ &= \frac{1}{2} \int x\sqrt{2x-3} \times 2 \, dx \\ &= \frac{1}{2} \int \frac{1}{2}(u+3)u^{1/2} \, du \\ &= \frac{1}{4} \int (u^{3/2} + 3u^{1/2}) \, du \\ &= \frac{1}{4} \left(\frac{2}{5}u^{5/2} + 2u^{3/2} \right) + c \\ &= \frac{1}{10}u^{3/2}(u+5) + c \\ &= \frac{1}{5}(x+1)(2x-3)^{3/2} + c. \end{aligned}$$

- (f) Let $u = x^2 - 4$; then $\frac{du}{dx} = 2x$. So

$$\begin{aligned} & \int x \cos(x^2 - 4) \, dx \\ &= \frac{1}{2} \int \cos(x^2 - 4)(2x) \, dx \\ &= \frac{1}{2} \int \cos u \, du \\ &= \frac{1}{2} \sin u + c \\ &= \frac{1}{2} \sin(x^2 - 4) + c. \end{aligned}$$

- (g) We use the hint given in the question:

$$\begin{aligned} & \int \frac{1}{e^{2x} + 1} \, dx \\ &= \int \frac{e^{2x} + 1 - e^{2x}}{e^{2x} + 1} \, dx \\ &= \int \left(1 - \frac{e^{2x}}{e^{2x} + 1} \right) \, dx \\ &= \int 1 \, dx - \int \frac{e^{2x}}{e^{2x} + 1} \, dx. \end{aligned}$$

The first integral is

$$\int 1 \, dx = x + c.$$

For the second integral, let $u = e^{2x} + 1$;

then $\frac{du}{dx} = 2e^{2x}$. So

$$\begin{aligned} & \int \frac{e^{2x}}{e^{2x} + 1} \, dx \\ &= \frac{1}{2} \int \frac{1}{e^{2x} + 1} (2e^{2x}) \, dx \\ &= \frac{1}{2} \int \frac{1}{u} \, du \\ &= \frac{1}{2} \ln |u| + c \\ &= \frac{1}{2} \ln(e^{2x} + 1) + c. \end{aligned}$$

Adding these integrals and continuing to use the symbol c for the resulting arbitrary constant gives

$$\int \frac{1}{e^{2x} + 1} \, dx = x - \frac{1}{2} \ln(e^{2x} + 1) + c.$$

Solution to Exercise 3

- (a) Let $u = 4 + \sin x$; then $\frac{du}{dx} = \cos x$.

Putting $x = 0$ gives $u = 4 + \sin 0 = 4$, and putting $x = \pi/2$ gives $u = 4 + \sin(\pi/2) = 5$. So

$$\begin{aligned} & \int_0^{\pi/2} \frac{\cos x}{4 + \sin x} \, dx \\ &= \int_0^{\pi/2} \left(\frac{1}{4 + \sin x} \right) \cos x \, dx \\ &= \int_4^5 \frac{1}{u} \, du \\ &= [\ln |u|]_4^5 \\ &= \ln 5 - \ln 4 \\ &= \ln \frac{5}{4}. \end{aligned}$$

- (b) Let $u = x + 2$; then $\frac{du}{dx} = 1$. Also,

$$x = u - 2, \text{ so } 2x + 1 = 2u - 3.$$

Putting $x = -2$ gives $u = 0$, and putting $x = -1$ gives $u = 1$. So

$$\begin{aligned} & \int_{-2}^{-1} (2x + 1)(x + 2)^3 \, dx \\ &= \int_0^1 (2u - 3)u^3 \, du \\ &= \int_0^1 (2u^4 - 3u^3) \, du \\ &= \left[\frac{2}{5}u^5 - \frac{3}{4}u^4 \right]_0^1 \\ &= \frac{2}{5} - \frac{3}{4} \\ &= -\frac{7}{20}. \end{aligned}$$

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- (c) Let $u = 3 + e^x$; then $\frac{du}{dx} = e^x$.

Putting $x = 0$ gives $u = 3 + e^0 = 4$, and putting $x = \ln 6$ gives $u = 3 + e^{\ln 6} = 9$.

So

$$\begin{aligned} & \int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}} dx \\ &= \int_0^{\ln 6} \left(\frac{1}{\sqrt{3+e^x}} \right) e^x dx \\ &= \int_4^9 \frac{1}{\sqrt{u}} du \\ &= \int_4^9 u^{-1/2} du \\ &= [2\sqrt{u}]_4^9 \\ &= (2\sqrt{9} - 2\sqrt{4}) \\ &= 2. \end{aligned}$$

Solution to Exercise 4

- (a) Integrating by parts gives

$$\begin{aligned} & \int 2x \sec^2 x dx \\ &= 2x \tan x - \int 2 \times \tan x dx \\ &= 2(x \tan x + \ln |\cos x|) + c. \end{aligned}$$

- (b) Integrating by parts twice gives

$$\begin{aligned} & \int x^2 e^{x/2} dx \\ &= x^2 \times 2e^{x/2} - \int 2x \times 2e^{x/2} dx \\ &= 2x^2 e^{x/2} - 4 \int x e^{x/2} dx \\ &= 2x^2 e^{x/2} - 4 \left(x \times 2e^{x/2} - \int 1 \times 2e^{x/2} dx \right) \\ &= 2x^2 e^{x/2} - 8xe^{x/2} + 16e^{x/2} + c \\ &= 2e^{x/2}(x^2 - 4x + 8) + c. \end{aligned}$$

- (c) Integrating by parts gives

$$\begin{aligned} & \int (x-1)(x+2)^4 dx \\ &= (x-1) \times \frac{1}{5}(x+2)^5 - \int 1 \times \frac{1}{5}(x+2)^5 dx \\ &= \frac{1}{5}(x-1)(x+2)^5 - \frac{1}{30}(x+2)^6 + c \\ &= \frac{1}{30}(x+2)^5(6(x-1) - (x+2)) + c \\ &= \frac{1}{30}(x+2)^5(5x-8) + c. \end{aligned}$$

- (d) We cannot integrate $\ln x$, so we must differentiate $\ln x$ and integrate x .

Integrating by parts gives

$$\begin{aligned} & \int x \ln x dx \\ &= \int (\ln x)x dx \\ &= \ln x \times \left(\frac{1}{2}x^2\right) - \int \frac{1}{x} \times \frac{1}{2}x^2 dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \left(\frac{1}{2}x^2\right) + c \\ &= \frac{1}{4}x^2(2 \ln x - 1) + c. \end{aligned}$$

- (e) We cannot integrate $\ln x$, so we differentiate $(\ln x)^2$ and integrate x^2 , and integrate by parts twice:

$$\begin{aligned} & \int x^2 (\ln x)^2 dx \\ &= \int (\ln x)^2 \times x^2 dx \\ &= (\ln x)^2 \times \frac{1}{3}x^3 \\ &\quad - \int 2(\ln x) \left(\frac{1}{x}\right) \times \frac{1}{3}x^3 dx \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3} \int (\ln x) \times x^2 dx \\ &= \frac{1}{3}x^3 (\ln x)^2 \\ &\quad - \frac{2}{3} \left(\ln x \times \frac{1}{3}x^3 - \int \frac{1}{x} \times \frac{1}{3}x^3 dx \right) \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{9} \int x^2 dx \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + c \\ &= \frac{1}{27}x^3 (9(\ln x)^2 - 6 \ln x + 2) + c. \end{aligned}$$

Solution to Exercise 5

Any of the three methods of finding partial fractions can be used here: equating coefficients, substituting values or the cover-up method. Each of these solutions shows a different method.

- (a) We have

$$\frac{4x-1}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4},$$

where A and B are constants.

Multiplying both sides by $(x+1)(x-4)$ and collecting terms gives

$$\begin{aligned} 4x-1 &= A(x-4) + B(x+1) \\ &= (A+B)x + (-4A+B). \end{aligned}$$

Comparing coefficients of the terms in x and the constant terms gives

$$\begin{aligned} A + B &= 4, \\ -4A + B &= -1. \end{aligned}$$

Subtracting the second equation from the first equation gives

$$5A = 5, \quad \text{so} \quad A = 1.$$

Substituting $A = 1$ into the first equation then gives

$$B = 4 - 1 = 3.$$

Hence

$$\frac{4x-1}{(x+1)(x-4)} = \frac{1}{x+1} + \frac{3}{x-4}.$$

(Check: when $x = 0$, LHS = $\frac{-1}{1 \times (-4)} = \frac{1}{4}$ and RHS = $\frac{1}{1} + \frac{3}{-4} = \frac{1}{4}$.)

(b) We have

$$\frac{x+5}{3(1-2x)(1+x)} = \frac{A}{1-2x} + \frac{B}{1+x},$$

where A and B are constants.

Multiplying both sides by $3(1-2x)(1+x)$ gives

$$x+5 = 3A(1+x) + 3B(1-2x).$$

Choosing $x = \frac{1}{2}$ gives

$$\frac{1}{2} + 5 = 3A \times (1 + \frac{1}{2}),$$

so

$$\frac{11}{2} = \frac{9}{2}A,$$

thus $A = \frac{11}{9}$.

Choosing $x = -1$ gives

$$-1 + 5 = 3B \times (1 - 2 \times (-1)),$$

so

$$4 = 9B,$$

thus $B = \frac{4}{9}$.

Hence

$$\frac{x+5}{3(1-2x)(1+x)} = \frac{11}{9(1-2x)} + \frac{4}{9(1+x)}.$$

(Check: when $x = 0$, LHS = $\frac{5}{3 \times 1 \times 1} = \frac{5}{3}$ and RHS = $\frac{11}{9 \times 1} + \frac{4}{9 \times 1} = \frac{15}{9} = \frac{5}{3}$.)

(c) We have

$$\frac{4x+8}{x(3x+2)(x-2)} = \frac{A}{x} + \frac{B}{3x+2} + \frac{C}{x-2},$$

where A , B and C are constants.

Using the cover-up method with $x = 0$ gives

$$A = \frac{4 \times 0 + 8}{(3 \times 0 + 2)(0 - 2)} = \frac{8}{2 \times (-2)} = -2.$$

Using the cover-up method with $x = -\frac{2}{3}$ gives

$$B = \frac{4 \times (-\frac{2}{3}) + 8}{-\frac{2}{3} \times ((-\frac{2}{3}) - 2)} = \frac{\frac{16}{3}}{\frac{16}{9}} = 3.$$

Using the cover-up method with $x = 2$ gives

$$C = \frac{4 \times 2 + 8}{2(3 \times 2 + 2)} = \frac{16}{2 \times 8} = 1.$$

Hence

$$\begin{aligned} &\frac{4x+8}{x(3x+2)(x-2)} \\ &= -\frac{2}{x} + \frac{3}{3x+2} + \frac{1}{x-2}. \end{aligned}$$

(Check: when $x = 1$,

LHS = $\frac{12}{1 \times 5 \times (-1)} = -\frac{12}{5}$ and

RHS = $-\frac{2}{1} + \frac{3}{5} + \frac{1}{-1} = -\frac{12}{5}$.)

(d) We have

$$\begin{aligned} &\frac{5x^2 + 3x + 4}{(x-1)(2x+1)(x+2)} \\ &= \frac{A}{x-1} + \frac{B}{2x+1} + \frac{C}{x+2}, \end{aligned}$$

where A , B and C are constants.

Using the cover-up method with $x = 1$ gives

$$A = \frac{5 \times 1^2 + 3 \times 1 + 4}{(2 \times 1 + 1)(1 + 2)} = \frac{12}{9} = \frac{4}{3}.$$

Using the cover-up method with $x = -\frac{1}{2}$ gives

$$\begin{aligned} B &= \frac{5 \times (-\frac{1}{2})^2 + 3 \times (-\frac{1}{2}) + 4}{(-\frac{1}{2} - 1)(-\frac{1}{2} + 2)} \\ &= \frac{\frac{15}{4}}{\frac{-9}{4}} = -\frac{5}{3}. \end{aligned}$$

Using the cover-up method with $x = -2$ gives

$$C = \frac{5 \times (-2)^2 + 3 \times (-2) + 4}{(-2 - 1)(2 \times (-2) + 1)} = \frac{18}{9} = 2.$$

Hence

$$\begin{aligned} &\frac{5x^2 + 3x + 4}{(x-1)(2x+1)(x+2)} \\ &= \frac{4}{3(x-1)} - \frac{5}{3(2x+1)} + \frac{2}{x+2}. \end{aligned}$$

(Check: when $x = 0$,
 $\text{LHS} = \frac{4}{-1 \times 1 \times 2} = -2$ and
 $\text{RHS} = \frac{4}{3 \times (-1)} - \frac{5}{3 \times 1} + \frac{2}{2} = -2$.)

Solution to Exercise 6

(a) We have

$$\frac{3x+2}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

where A , B and C are constants.

Using the cover-up method with $x = 1$ gives

$$A = \frac{3 \times 1 + 2}{(1+2)^2} = \frac{5}{9}.$$

Using the cover-up method with $x = -2$ gives

$$C = \frac{3 \times (-2) + 2}{(-2-1)} = \frac{4}{3}.$$

Hence

$$\frac{3x+2}{(x-1)(x+2)^2} = \frac{5}{9(x-1)} + \frac{B}{x+2} + \frac{4}{3(x+2)^2}.$$

Multiplying both sides by

$9(x-1)(x+2)^2$ gives

$$\begin{aligned} 9(3x+2) &= 5(x+2)^2 + 9B(x+2)(x-1) + 12(x-1). \end{aligned}$$

Substituting $x = 0$ gives

$$18 = 5 \times 2^2 + 9B \times 2 \times (-1) + 12 \times (-1),$$

so

$$18 = 20 - 18B - 12,$$

thus

$$10 = -18B.$$

So $B = -\frac{5}{9}$. Therefore

$$\frac{3x+2}{(x-1)(x+2)^2} = \frac{5}{9(x-1)} - \frac{5}{9(x+2)} + \frac{4}{3(x+2)^2}.$$

(Check: when $x = 0$, $\text{LHS} = \frac{2}{-1 \times 2^2} = -\frac{1}{2}$
 and $\text{RHS} = \frac{5}{-9} - \frac{5}{18} + \frac{4}{12} = -\frac{9}{18} = -\frac{1}{2}$.)

(b) We have

$$\frac{x+1}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2},$$

where A and B are constants.

Using the cover-up method with $x = -3$ gives

$$B = \frac{-3+1}{1} = -2.$$

So

$$\frac{x+1}{(x+3)^2} = \frac{A}{x+3} - \frac{2}{(x+3)^2}.$$

Multiplying both sides by $(x+3)^2$ gives

$$x+1 = A(x+3) - 2.$$

Substituting $x = 0$ gives

$$1 = 3A - 2, \quad \text{so} \quad A = 1.$$

Therefore

$$\frac{x+1}{(x+3)^2} = \frac{1}{x+3} - \frac{2}{(x+3)^2}.$$

(Check: when $x = 0$, $\text{LHS} = \frac{1}{3^2} = \frac{1}{9}$ and
 $\text{RHS} = \frac{1}{3} - \frac{2}{3^2} = \frac{1}{9}$.)

(c) We have

$$\frac{2x^2+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3},$$

where A , B and C are constants.

Using the cover-up method with $x = -1$ gives

$$C = \frac{2 \times (-1)^2 + 3}{1} = 5.$$

Therefore

$$\frac{2x^2+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{5}{(x+1)^3}.$$

Multiplying both sides by $(x+1)^3$ gives

$$2x^2+3 = A(x+1)^2 + B(x+1) + 5.$$

Substituting $x = 0$ gives

$$3 = A + B + 5, \quad \text{so} \quad A + B = -2.$$

Substituting $x = -2$ gives

$$11 = A - B + 5, \quad \text{so} \quad A - B = 6.$$

Adding these two simultaneous equations gives

$$2A = 4, \quad \text{so} \quad A = 2 \quad \text{and} \quad B = -4.$$

Therefore

$$\frac{2x^2+3}{(x+1)^3} = \frac{2}{x+1} - \frac{4}{(x+1)^2} + \frac{5}{(x+1)^3}.$$

(Check: when $x = 0$, $\text{LHS} = \frac{3}{1^3} = 3$ and
 $\text{RHS} = \frac{2}{1} - \frac{4}{1^2} + \frac{5}{1^3} = 3$.)

Solution to Exercise 7

(a) Since

$$(-3)^2 - 4 \times 1 \times 4 = -7 < 0,$$

the quadratic expression $x^2 - 3x + 4$ cannot be factorised. So

$$\frac{x+3}{(x-1)(x^2-3x+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2-3x+4},$$

where A , B and C are constants.

Using the cover-up method with $x = 1$ gives

$$A = \frac{1+3}{1^2-3 \times 1+4} = 2.$$

So

$$\frac{x+3}{(x-1)(x^2-3x+4)} = \frac{2}{x-1} + \frac{Bx+C}{x^2-3x+4}.$$

Multiplying both sides by $(x-1)(x^2-3x+4)$ gives

$$x+3 = 2(x^2-3x+4) + (Bx+C)(x-1).$$

Substituting $x = 0$ gives

$$3 = 8 + C \times (-1), \quad \text{so } C = 5.$$

Substituting $x = 2$ and $C = 5$ gives

$$5 = 2(2^2 - 3 \times 2 + 4) + (2B + 5) \\ = 4 + 2B + 5,$$

so $2B = -4$ thus $B = -2$.

Therefore

$$\frac{x+3}{(x-1)(x^2-3x+4)} = \frac{2}{x-1} + \frac{5-2x}{x^2-3x+4}.$$

(Check: when $x = 0$, LHS = $\frac{3}{-1 \times 4} = -\frac{3}{4}$ and RHS = $\frac{2}{-1} - \frac{5}{4} = -\frac{3}{4}$.)

(b) Since

$$(0)^2 - 4 \times 1 \times 2 = -8 < 0,$$

the quadratic expression $x^2 + 2$ cannot be factorised. So

$$\frac{2}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2},$$

where A , B and C are constants.

Using the cover-up method with $x = 3$ gives

$$A = \frac{2}{3^2+2} = \frac{2}{11}.$$

So

$$\frac{2}{(x-3)(x^2+2)} = \frac{2}{11(x-3)} + \frac{Bx+C}{x^2+2}.$$

Multiplying both sides by

$11(x-3)(x^2+2)$ gives

$$22 = 2(x^2+2) + 11(Bx+C)(x-3).$$

Substituting $x = 0$ gives

$$22 = 2 \times 2 + 11C \times (-3) \\ = 4 - 33C,$$

so $C = -\frac{18}{33} = -\frac{6}{11}$.

Substituting $x = 1$ and $C = -\frac{6}{11}$ gives

$$22 = 2 \times 3 + 11 \left(B + \left(-\frac{6}{11} \right) \right) \times (-2) \\ = 6 - 22B + 12,$$

so $B = -\frac{4}{22} = -\frac{2}{11}$.

Therefore

$$\frac{2}{(x-3)(x^2+2)} = \frac{2}{11(x-3)} - \frac{2x+6}{11(x^2+2)}.$$

(Check: when $x = 0$, LHS = $\frac{2}{-3 \times 2} = -\frac{1}{3}$

and RHS = $\frac{2}{11 \times (-3)} - \frac{6}{11 \times 2} = -\frac{22}{66} = -\frac{1}{3}$.)

(c) Since

$$3^2 - 4 \times 2 \times 2 = -7 < 0,$$

the quadratic expression $2x^2 + 3x + 2$ cannot be factorised. So

$$\frac{x^2+x+1}{x(2x^2+3x+2)} = \frac{A}{x} + \frac{Bx+C}{2x^2+3x+2},$$

where A , B and C are constants.

Using the cover-up method with $x = 0$ gives $A = \frac{1}{2}$. So

$$\frac{x^2+x+1}{x(2x^2+3x+2)} = \frac{1}{2x} + \frac{Bx+C}{2x^2+3x+2}.$$

Multiplying both sides by

$2x(2x^2+3x+2)$ and collecting terms gives

$$2(x^2+x+1) = 2x^2+3x+2 + 2x(Bx+C),$$

so

$$2x^2+2x+2 = (2+2B)x^2 + (2C+3)x + 2.$$

Comparing coefficients of x^2 gives

$$2 = 2 + 2B, \quad \text{so } B = 0.$$

Exercise Booklet 7

Comparing coefficients of x gives

$$2 = 2C + 3, \quad \text{so} \quad C = -\frac{1}{2}.$$

Therefore

$$\frac{x^2 + x + 1}{x(2x^2 + 3x + 2)} = \frac{1}{2x} - \frac{1}{2(2x^2 + 3x + 2)}.$$

(Check: when $x = 1$, LHS = $\frac{3}{7}$ and
RHS = $\frac{1}{2} - \frac{1}{14} = \frac{6}{14} = \frac{3}{7}$.)

Solution to Exercise 8

(a) We have

$$\begin{aligned} \frac{2x-7}{x^2-x-2} &= \frac{2x-7}{(x-2)(x+1)} \\ &= \frac{A}{x-2} + \frac{B}{x+1}, \end{aligned}$$

where A and B are constants.

Using the cover-up method with $x = 2$ gives

$$A = \frac{2 \times 2 - 7}{2 + 1} = -1.$$

Using the cover-up method with $x = -1$ gives

$$B = \frac{2 \times (-1) - 7}{-1 - 2} = \frac{-9}{-3} = 3.$$

So

$$\frac{2x-7}{x^2-x-2} = -\frac{1}{x-2} + \frac{3}{x+1}.$$

(Check: when $x = 0$, LHS = $\frac{-7}{-2} = \frac{7}{2}$ and
RHS = $-\frac{1}{-2} + \frac{3}{1} = \frac{1}{2} + 3 = \frac{7}{2}$.)

So

$$\begin{aligned} &\int \frac{2x-7}{(x-2)(x+1)} dx \\ &= \int \left(-\frac{1}{x-2} + \frac{3}{x+1} \right) dx \\ &= -\int \frac{1}{x-2} dx + 3 \int \frac{1}{x+1} dx \\ &= -\ln|x-2| + 3\ln|x+1| + c \\ &= \ln \left| \frac{(x+1)^3}{x-2} \right| + c. \end{aligned}$$

(b) We have

$$\begin{aligned} \frac{x^2-3x-9}{x^3-6x^2+9x} &= \frac{x^2-3x-9}{x(x-3)^2} \\ &= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}, \end{aligned}$$

where A , B and C are constants.

Using the cover-up method with $x = 0$ gives

$$A = \frac{-9}{(-3)^2} = -1.$$

Using the cover-up method with $x = 3$ gives

$$C = \frac{3^2 - 3 \times 3 - 9}{3} = -3.$$

So

$$\frac{x^2-3x-9}{x(x-3)^2} = -\frac{1}{x} + \frac{B}{x-3} - \frac{3}{(x-3)^2}.$$

Multiplying both sides by $x(x-3)^2$ gives

$$x^2 - 3x - 9 = -(x-3)^2 + Bx(x-3) - 3x.$$

Substituting $x = 1$ gives

$$1^2 - 3 \times 1 - 9 = -(-2)^2 + B(-2) - 3 \times 1,$$

so

$$-11 = -4 - 2B - 3,$$

thus

$$-4 = -2B,$$

hence $B = 2$.

Therefore

$$\frac{x^2-3x-9}{x^3-6x^2+9x} = -\frac{1}{x} + \frac{2}{x-3} - \frac{3}{(x-3)^2}.$$

(Check: when $x = 1$, LHS = $-\frac{11}{4}$ and
RHS = $-\frac{1}{1} + \frac{2}{-2} - \frac{3}{4} = -2 - \frac{3}{4} = -\frac{11}{4}$.)

So

$$\begin{aligned} &\int \frac{x^2-3x-9}{x(x-3)^2} dx \\ &= \int \left(-\frac{1}{x} + \frac{2}{x-3} - \frac{3}{(x-3)^2} \right) dx \\ &= -\ln|x| + 2\ln|x-3| + \frac{3}{x-3} + c \\ &= \ln \left| \frac{(x-3)^2}{x} \right| + \frac{3}{x-3} + c. \end{aligned}$$

(c) The expression $x^2 + 9$ cannot be factorised. So we have

$$\frac{6x^2 - x + 30}{2(2x+3)(x^2+9)} = \frac{A}{2x+3} + \frac{Bx+C}{x^2+9},$$

where A , B and C are constants.

Using the cover-up method with $x = -\frac{3}{2}$ gives

$$\begin{aligned} A &= \frac{6\left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right) + 30}{2\left(\left(-\frac{3}{2}\right)^2 + 9\right)} \\ &= \frac{\frac{27}{2} + \frac{3}{2} + 30}{2\left(\frac{9}{4} + 9\right)} \\ &= \frac{15 + 30}{2 \times \frac{45}{4}} \\ &= \frac{45}{\left(\frac{45}{2}\right)} \\ &= 2. \end{aligned}$$

So

$$\frac{6x^2 - x + 30}{2(2x + 3)(x^2 + 9)} = \frac{2}{2x + 3} + \frac{Bx + C}{x^2 + 9}.$$

Multiplying both sides by $2(2x + 3)(x^2 + 9)$ gives

$$\begin{aligned} 6x^2 - x + 30 \\ &= 4(x^2 + 9) + 2(Bx + C)(2x + 3). \end{aligned}$$

Comparing coefficients of x^2 gives

$$6 = 4 + 4B,$$

so $4B = 2$ and hence $B = \frac{1}{2}$. Comparing constant terms gives

$$30 = 36 + 6C,$$

so $6C = -6$ and hence $C = -1$. Hence

$$\frac{6x^2 - x + 30}{2(2x + 3)(x^2 + 9)} = \frac{2}{2x + 3} + \frac{\frac{1}{2}x - 1}{x^2 + 9},$$

that is,

$$\frac{6x^2 - x + 30}{2(2x + 3)(x^2 + 9)} = \frac{2}{2x + 3} + \frac{x - 2}{2(x^2 + 9)}.$$

(Check: when $x = 0$, LHS = $\frac{30}{2 \times 3 \times 9} = \frac{5}{9}$ and RHS = $\frac{2}{3} + \frac{-2}{18} = \frac{6}{9} - \frac{1}{9} = \frac{5}{9}$.)

So

$$\begin{aligned} &\int \frac{6x^2 - x + 30}{2(2x + 3)(x^2 + 9)} dx \\ &= \int \frac{2}{2x + 3} dx + \int \frac{x}{2(x^2 + 9)} dx \\ &\quad - \int \frac{1}{x^2 + 9} dx. \end{aligned}$$

The first integral on the right-hand side is

$$\begin{aligned} \int \frac{2}{2x + 3} dx &= 2 \times \frac{1}{2} \ln |2x + 3| + c \\ &= \ln |2x + 3| + c. \end{aligned}$$

For the second integral on the right-hand side, let $u = x^2 + 9$; then $\frac{du}{dx} = 2x$. So

$$\begin{aligned} \int \frac{x}{2(x^2 + 9)} dx &= \frac{1}{4} \int \frac{2x}{x^2 + 9} dx \\ &= \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln |u| + c \\ &= \frac{1}{4} \ln |x^2 + 9| + c \\ &= \frac{1}{4} \ln(x^2 + 9) + c. \end{aligned}$$

The third integral on the right-hand side is

$$\begin{aligned} - \int \frac{1}{x^2 + 9} dx &= -\frac{1}{9} \int \frac{1}{\left(\frac{x}{3}\right)^2 + 1} dx \\ &= -\frac{1}{9} \times 3 \tan^{-1} \left(\frac{x}{3}\right) + c \\ &= -\frac{1}{3} \tan^{-1} \left(\frac{x}{3}\right) + c. \end{aligned}$$

Adding these three integrals, and continuing to use the symbol c for the resulting arbitrary constant, gives

$$\begin{aligned} &\int \frac{6x^2 - x + 30}{2(2x + 3)(x^2 + 9)} dx \\ &= \ln |2x + 3| + \frac{1}{4} \ln(x^2 + 9) \\ &\quad - \frac{1}{3} \tan^{-1} \left(\frac{x}{3}\right) + c \\ &= \frac{1}{4} \ln(2x + 3)^4 + \frac{1}{4} \ln(x^2 + 9) \\ &\quad - \frac{1}{3} \tan^{-1} \left(\frac{x}{3}\right) + c \\ &= \frac{1}{4} \ln((2x + 3)^4(x^2 + 9)) \\ &\quad - \frac{1}{3} \tan^{-1} \left(\frac{x}{3}\right) + c. \end{aligned}$$

Solution to Exercise 9

$$\begin{array}{r} \text{(a)} \qquad \qquad \qquad 4x^2 - 5x + 3 \\ 2x + 1 \overline{) 8x^3 - 6x^2 + x + 5} \\ \underline{8x^3 + 4x^2} \\ -10x^2 + x + 5 \\ \underline{-10x^2 - 5x} \\ 6x + 5 \\ \underline{6x + 3} \\ 2 \end{array}$$

So the quotient on dividing $8x^3 - 6x^2 + x + 5$ by $2x + 1$ is $4x^2 - 5x + 3$, and the remainder is 2.

Exercise Booklet 7

(Check:

$$\begin{aligned} & (4x^2 - 5x + 3)(2x + 1) + 2 \\ &= 8x^3 + 4x^2 - 10x^2 - 5x + 6x + 3 + 2 \\ &= 8x^3 - 6x^2 + x + 5.) \end{aligned}$$

Therefore

$$\frac{8x^3 - 6x^2 + x + 5}{2x + 1} = 4x^2 - 5x + 3 + \frac{2}{2x + 1}.$$

(b)

$$\begin{array}{r} 2x + 3 \\ x^2 + 0x + 1 \overline{) 2x^3 + 3x^2 - 3x + 1} \\ \underline{2x^3 + 0x^2 + 2x} \\ 3x^2 - 5x + 1 \\ \underline{3x^2 + 0x + 3} \\ -5x - 2 \end{array}$$

So the quotient on dividing

$2x^3 + 3x^2 - 3x + 1$ by $x^2 + 1$ is $2x + 3$,
and the remainder is $-5x - 2$.

(Check:

$$\begin{aligned} & (2x+3)(x^2+1)+(-5x-2) \\ &= 2x^3+2x+3x^2+3-5x-2 \\ &= 2x^3+3x^2-3x+1.) \end{aligned}$$

Therefore

$$\frac{2x^3 + 3x^2 - 3x + 1}{x^2 + 1} = 2x + 3 - \frac{5x + 2}{x^2 + 1}.$$

(c)

$$\begin{array}{r}
 4x^2 + 2x - 7 \\
 2x^2 - x + 4 \overline{) 8x^4 + 0x^3 + 0x^2 + 15x - 30} \\
 \underline{8x^4 - 4x^3 + 16x^2} \\
 4x^3 - 16x^2 + 15x - 30 \\
 \underline{4x^3 - 2x^2 + 8x} \\
 -14x^2 + 7x - 30 \\
 \underline{-14x^2 + 7x - 28} \\
 -2
 \end{array}$$

So the quotient on dividing

$8x^4 + 15x - 30$ by $2x^2 - x + 4$ is
 $4x^2 + 2x - 7$, and the remainder is -2 .

(Check:

$$\begin{aligned} & (4x^2 + 2x - 7)(2x^2 - x + 4) - 2 \\ &= 8x^4 - 4x^3 + 16x^2 + 4x^3 - 2x^2 + 8x \\ &\quad - 14x^2 + 7x - 28 - 2 \\ &= 8x^4 + 15x - 30.) \end{aligned}$$

Therefore

$$\frac{8x^4 + 15x - 30}{2x^2 - x + 4}$$
$$= 4x^2 + 2x - 7 - \frac{2}{2x^2 - x + 4}.$$

Solution to Exercise 10

(a) We have

$$\frac{x^2 - 4x + 1}{x(x + 1)} = \frac{x^2 - 4x + 1}{x^2 + x}.$$

$$\begin{array}{r} 1 \\ x^2 + x \overline{) x^2 - 4x + 1} \\ \underline{x^2 + x} \\ -5x + 1 \end{array}$$

So the quotient on dividing $x^2 - 4x + 1$ by $x(x + 1)$ is 1, and the remainder is $-5x + 1$.

Therefore

$$\frac{x^2 - 4x + 1}{x(x + 1)} = 1 + \frac{1 - 5x}{x(x + 1)}.$$

Now

$$\frac{1-5x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1},$$

where A and B are constants.

Using the cover-up method with $x = 0$ gives $A = 1$, and using the cover-up method with $x = -1$ gives $B = -6$. So

$$\frac{1-5x}{x(x+1)} = \frac{1}{x} - \frac{6}{x+1}.$$

Hence

$$\frac{x^2 - 4x + 1}{x(x + 1)} = 1 + \frac{1}{x} - \frac{6}{x + 1}.$$

(Check: when $x = 1$, LHS = $\frac{-2}{2} = -1$
and RHS = $1 + 1 - \frac{6}{2} = -1$.)

(b)

$$x^2 + 3x - 4 \overline{\begin{array}{r} x \\ x^3 + 3x^2 - 3x - 11 \\ \underline{x^3 + 3x^2 - 4x} \end{array}}$$

So the quotient on dividing $x^3 + 3x^2 - 3x - 11$ by $x^2 + 3x - 4$ is x , and the remainder is $x - 11$.

Therefore

$$\frac{x^3 + 3x^2 - 3x - 11}{x^2 + 3x - 4} = x + \frac{x - 11}{x^2 + 3x - 4}.$$

Now

$$\begin{aligned}\frac{x - 11}{x^2 + 3x - 4} &= \frac{x - 11}{(x + 4)(x - 1)} \\ &= \frac{A}{x + 4} + \frac{B}{x - 1},\end{aligned}$$

where A and B are constants.

Using the cover-up method with $x = -4$ gives

$$A = \frac{-4 - 11}{-4 - 1} = 3.$$

Using the cover-up method with $x = 1$ gives

$$B = \frac{1 - 11}{1 + 4} = -2.$$

So

$$\frac{x - 11}{x^2 + 3x - 4} = \frac{3}{x + 4} - \frac{2}{x - 1}.$$

Hence

$$\frac{x^3 + 3x^2 - 3x - 11}{x^2 + 3x - 4} = x + \frac{3}{x + 4} - \frac{2}{x - 1}.$$

(Check: when $x = 0$, LHS = $\frac{-11}{-4} = \frac{11}{4}$ and RHS = $\frac{3}{4} - \frac{2}{-1} = \frac{3}{4} + 2 = \frac{11}{4}$.)

Solution to Exercise 11

(a) We have

$$\frac{x^2 + 1}{x(x + 3)} = \frac{x^2 + 1}{x^2 + 3x}.$$

$$\begin{array}{r} x^2 + 3x \overline{) x^2 + 0x + 1} \\ \underline{x^2 + 3x} \\ -3x + 1 \end{array}$$

So the quotient on dividing $x^2 + 1$ by $x(x + 3)$ is 1, and the remainder is $-3x + 1$. Therefore

$$\frac{x^2 + 1}{x(x + 3)} = 1 + \frac{1 - 3x}{x(x + 3)}.$$

Now

$$\frac{1 - 3x}{x(x + 3)} = \frac{A}{x} + \frac{B}{x + 3},$$

where A and B are constants.

Using the cover-up method with $x = 0$ gives $A = \frac{1}{3}$.

Using the cover-up method with $x = -3$ gives

$$B = \frac{1 - 3 \times (-3)}{-3} = -\frac{10}{3}.$$

So

$$\frac{1 - 3x}{x(x + 3)} = \frac{1}{3x} - \frac{10}{3(x + 3)}.$$

Hence

$$\frac{x^2 + 1}{x(x + 3)} = 1 + \frac{1}{3x} - \frac{10}{3(x + 3)}.$$

(Check: when $x = 1$, LHS = $\frac{2}{4} = \frac{1}{2}$ and RHS = $1 + \frac{1}{3} - \frac{10}{3 \times 4} = \frac{3}{6} = \frac{1}{2}$.)

So

$$\begin{aligned}\int \frac{x^2 + 1}{x(x + 3)} dx &= \int \left(1 + \frac{1}{3x} - \frac{10}{3(x + 3)} \right) dx \\ &= x + \frac{1}{3} \ln |x| - \frac{10}{3} \ln |x + 3| + c \\ &= x + \frac{1}{3} \ln \left| \frac{x}{(x + 3)^{10}} \right| + c.\end{aligned}$$

(b)

$$\begin{array}{r} x^2 + 2x + 3 \overline{) x^4 + 0x^3 + 0x^2 + 0x + 0} \\ \underline{x^4 - 2x^3 + x^2} \\ 2x^3 - x^2 \\ \underline{2x^3 - 4x^2 + 2x} \\ 3x^2 - 2x \\ \underline{3x^2 - 6x + 3} \\ 4x - 3 \end{array}$$

So the quotient on dividing x^4 by $x^2 - 2x + 1$ is $x^2 + 2x + 3$, and the remainder is $4x - 3$. Therefore

$$\frac{x^4}{x^2 - 2x + 1} = x^2 + 2x + 3 + \frac{4x - 3}{x^2 - 2x + 1}.$$

Now

$$\begin{aligned}\frac{4x - 3}{x^2 - 2x + 1} &= \frac{4x - 3}{(x - 1)^2} \\ &= \frac{A}{x - 1} + \frac{B}{(x - 1)^2},\end{aligned}$$

where A and B are constants.

Exercise Booklet 7

Using the cover-up method with $x = 1$ gives

$$B = \frac{4 \times 1 - 3}{1} = 1.$$

So

$$\frac{4x - 3}{(x - 1)^2} = \frac{A}{x - 1} + \frac{1}{(x - 1)^2}.$$

Multiplying both sides by $(x - 1)^2$ gives

$$4x - 3 = A(x - 1) + 1.$$

Substituting $x = 0$ gives

$$-3 = -A + 1, \quad \text{so} \quad A = 4.$$

Hence

$$\frac{4x - 3}{(x - 1)^2} = \frac{4}{x - 1} + \frac{1}{(x - 1)^2}.$$

So

$$\begin{aligned} \frac{x^4}{x^2 - 2x + 1} &= x^2 + 2x + 3 \\ &\quad + \frac{4}{x - 1} + \frac{1}{(x - 1)^2}. \end{aligned}$$

(Check: when $x = 0$, LHS = 0 and
RHS = $3 + \frac{4}{-1} + \frac{1}{(-1)^2} = 3 - 4 + 1 = 0$.)

Therefore

$$\begin{aligned} &\int \frac{x^4}{x^2 - 2x + 1} dx \\ &= \int \left(x^2 + 2x + 3 + \frac{4}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \frac{1}{3}x^3 + x^2 + 3x + 4 \ln|x - 1| - \frac{1}{x - 1} + c. \end{aligned}$$

Solution to Exercise 12

- (a) The denominator $x + 3$ of $k(x)$ is 0 when $x = -3$, so that value must be excluded from the domain. Therefore the domain of k is

$$(-\infty, -3) \cup (-3, \infty).$$

- (b) The denominator $(x + 2)^2$ of $f(x)$ is 0 when $x = -2$, so the domain of f is

$$(-\infty, -2) \cup (-2, \infty).$$

- (c) We have

$$g(x) = \frac{x}{x^2 - 3x - 4} = \frac{x}{(x - 4)(x + 1)}.$$

The denominator of $g(x)$ is 0 when $x = 4$ and $x = -1$, so these values must be excluded from the domain. Therefore the domain of g is

$$(-\infty, -1) \cup (-1, 4) \cup (4, \infty).$$

- (d) We have

$$h(x) = \frac{3}{2x^2 + 3x} = \frac{3}{x(2x + 3)}.$$

The denominator of $h(x)$ is 0 when $x = 0$ and $x = -\frac{3}{2}$, so the domain of h is

$$(-\infty, -\frac{3}{2}) \cup (-\frac{3}{2}, 0) \cup (0, \infty).$$

Solution to Exercise 13

- (a) The y -intercept is $f(0) = \frac{3}{4}$.

The equation $f(x) = 0$ has no solutions, so there are no x -intercepts.

- (b) The y -intercept is $g(0) = 0$.

The x -intercepts are the solutions of $g(x) = 0$. Multiplying both sides of this equation by $x^2 - 3x - 4$ gives $x = 0$. So the only x -intercept is 0.

- (c) Since $x = 0$ is not in the domain of $h(x)$, there is no y -intercept.

The equation $h(x) = 0$ has no solutions, so there are no x -intercepts.

- (d) The y -intercept is $k(0) = \frac{7}{3}$.

The x -intercepts are the solutions of $k(x) = 0$. Multiplying both sides of this equation by $x + 3$ gives

$$x^2 + 7 = 0.$$

This equation has no solutions, so there are no x -intercepts.

- (e) The y -intercept is $l(0) = -\frac{5}{3}$.

The x -intercepts are the solutions of $l(x) = 0$. Multiplying both sides of this equation by $3 - x$ gives $2x - 5 = 0$, so the x -intercept is $x = \frac{5}{2}$.

Solution to Exercise 14

- (a) By the quotient rule,

$$\begin{aligned} k'(x) &= \frac{(x + 3)(2x) - (x^2 + 7) \times 1}{(x + 3)^2} \\ &= \frac{x^2 + 6x - 7}{(x + 3)^2} \\ &= \frac{(x + 7)(x - 1)}{(x + 3)^2}. \end{aligned}$$

Multiplying both sides of the equation $k'(x) = 0$ by $(x + 3)^2$ gives

$$(x + 7)(x - 1) = 0.$$

So the stationary points of k are -7 and 1 .

Since

$$k(-7) = \frac{(-7)^2 + 7}{-7 + 3} = -14$$

and

$$k(1) = \frac{1^2 + 7}{1 + 3} = 2,$$

the coordinates of the stationary points are $(-7, -14)$ and $(1, 2)$.

- (b) For reasons of space, the following table of signs for $k'(x)$ is split into two parts.

x	$(-\infty, -7)$	-7	$(-7, -3)$	-3
$(x + 7)$	$-$	0	$+$	$+$
$(x - 1)$	$-$	$-$	$-$	$-$
$(x + 3)^2$	$+$	$+$	$+$	0
$k'(x)$	$+$	0	$-$	$*$

x	$(-3, 1)$	1	$(1, \infty)$
$(x + 7)$	$+$	$+$	$+$
$(x - 1)$	$-$	0	$+$
$(x + 3)^2$	$+$	$+$	$+$
$k'(x)$	$-$	0	$+$

Hence k is increasing on the intervals $(-\infty, -7)$ and $(1, \infty)$, and k is decreasing on the intervals $(-7, -3)$ and $(-3, 1)$.

- (c) Since k is increasing to the left and decreasing to the right of the stationary point -7 , we deduce that -7 is a local maximum of k .

Since k is decreasing to the left and increasing to the right of the stationary point 1 , we deduce that 1 is a local minimum of k .

Solution to Exercise 15

Vertical asymptotes of a rational function f occur at values of x at which the denominator of $f(x)$ is zero.

- (a) The denominator $2x^2$ of $p(x)$ is 0 when $x = 0$, so the line $x = 0$ is a vertical asymptote of the graph of p .

We have

$$p(x) = \frac{3}{2x^2} = \frac{3}{2}x^{-2},$$

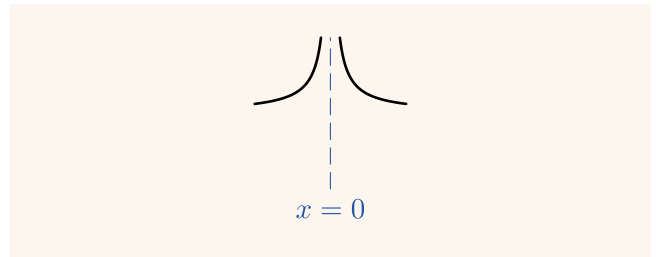
so

$$p'(x) = -3x^{-3} = -\frac{3}{x^3}.$$

To the left of $x = 0$, the value of $p'(x)$ is positive, so p is increasing.

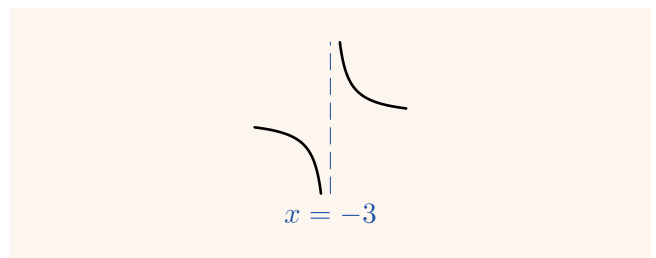
To the right of $x = 0$, the value of $p'(x)$ is negative, so p is decreasing.

Therefore the behaviour of the graph of p near $x = 0$ is as sketched below.



- (b) The denominator $x + 3$ of $k(x)$ is 0 when $x = -3$, so the line $x = -3$ is a vertical asymptote of the graph of k .

From the table of signs in the solution to Exercise 14(b), we see that k is decreasing to the left and right of $x = -3$, so the behaviour of the graph of k near $x = -3$ is as sketched below.

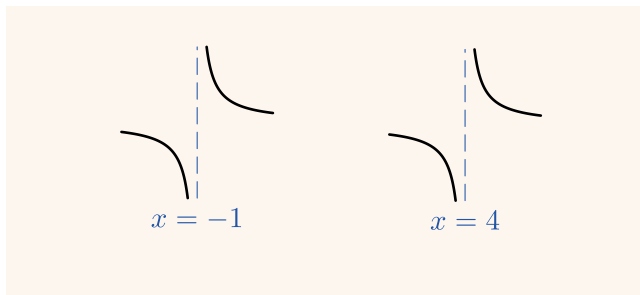


- (c) The denominator of $g(x)$ is $x^2 - 3x - 4 = (x - 4)(x + 1)$ (as in Exercise 12(c)), which is 0 when $x = -1$ and $x = 4$, so the lines $x = -1$ and $x = 4$ are vertical asymptotes of the graph of g .

By the quotient rule,

$$\begin{aligned} g'(x) &= \frac{(x^2 - 3x - 4) \times 1 - x(2x - 3)}{(x^2 - 3x - 4)^2} \\ &= -\frac{x^2 + 4}{(x^2 - 3x - 4)^2}. \end{aligned}$$

The expressions $x^2 + 4$ and $(x^2 - 3x - 4)^2$ are both positive for all values of x in the domain of g , so $g'(x)$ is negative and g is decreasing on either side of each of the vertical asymptotes. Therefore the behaviour of the graph of g near $x = -1$ and $x = 4$ is as sketched below.



Solution to Exercise 16

- (a) Ignoring all terms other than the dominant terms in the numerator and denominator of $f(x)$ gives

$$g(x) = \frac{-2x^2}{x^2} = -2.$$

Since $f(x)$ has the same asymptotic behaviour as $x \rightarrow \infty$ and as $x \rightarrow -\infty$ as $g(x)$, we see that $f(x) \rightarrow -2$ as $x \rightarrow \infty$, and $f(x) \rightarrow -2$ as $x \rightarrow -\infty$, and the graph of f has a horizontal asymptote with equation $y = -2$.

- (b) The degree of the numerator of $f(x)$ is one more than the degree of the denominator, so the graph of f has a slant asymptote.

We divide the numerator of $f(x)$ by its denominator using polynomial long division.

$$\begin{array}{r} 2x \\ 2x^2 + 5 \overline{) 4x^3 + 0x^2 + 0x - 1} \\ \underline{4x^3 + 0x^2 + 10x} \\ -10x - 1 \end{array}$$

(Check:

$$\begin{aligned} (2x)(2x^2 + 5) + (-10x - 1) \\ = 4x^3 + 10x - 10x - 1 \\ = 4x^3 - 1. \end{aligned}$$

It follows that

$$f(x) = 2x - \frac{10x + 1}{2x^2 + 5}.$$

The proper rational expression here tends to 0 as $x \rightarrow \infty$ and as $x \rightarrow -\infty$. So we see that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, and the graph of f has a slant asymptote with equation $y = 2x$.

- (c) The degree of the numerator of $f(x)$ is one more than the degree of the

denominator, so the graph of f has a slant asymptote.

We divide the numerator of $f(x)$ by its denominator using polynomial long division.

$$\begin{array}{r} x + 2 \\ 8x^3 + 27 \overline{) 8x^4 + 16x^3 + 0x^2 + 7x - 11} \\ \underline{8x^4 + 0x^3 + 0x^2 + 27x} \\ 16x^3 + 0x^2 - 20x - 11 \\ \underline{16x^3 + 0x^2 + 0x + 54} \\ -20x - 65 \end{array}$$

(Check:

$$\begin{aligned} (x + 2)(8x^3 + 27) + (-20x - 65) \\ = 8x^4 + 27x + 16x^3 + 54 - 20x - 65 \\ = 8x^4 + 16x^3 + 7x - 11. \end{aligned}$$

It follows that

$$f(x) = x + 2 - \frac{20x + 65}{8x^3 + 27}.$$

The proper rational expression here tends to 0 as $x \rightarrow \infty$ and as $x \rightarrow -\infty$. So we see that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, and the graph of f has a slant asymptote with equation $y = x + 2$.

Solution to Exercise 17

- (a) Since $f(1) = \frac{3}{2}$ and $f(-1) = -\frac{1}{2}$, f is neither an even function nor an odd function.

- (b) Since

$$\begin{aligned} f(-x) &= \frac{2}{-x} + (-x) \\ &= -\frac{2}{x} - x \\ &= -\left(\frac{2}{x} + x\right) \\ &= -f(x), \end{aligned}$$

f is an odd function.

- (c) Since

$$\begin{aligned} f(-x) &= \frac{2(-x)^4 - (-x)^2 + 1}{(-x)^2 + 5} \\ &= \frac{2x^4 - x^2 + 1}{x^2 + 5} \\ &= f(x), \end{aligned}$$

f is an even function.

Solution to Exercise 18

The steps in each solution correspond to those in the graph-sketching strategy.

(a) The function is

$$f(x) = \frac{5}{(x-3)^2}.$$

1. The denominator $(x-3)^2$ of $f(x)$ is 0 only when $x = 3$, so the domain of f is $(-\infty, 3) \cup (3, \infty)$. Therefore the line $x = 3$ is a vertical asymptote of the graph of f .

2. The y -intercept is

$$f(0) = \frac{5}{(-3)^2} = \frac{5}{9}.$$

The equation $f(x) = 0$ is

$$\frac{5}{(x-3)^2} = 0,$$

which has no solutions. Hence there are no x -intercepts.

3. We have

$$f(x) = \frac{5}{(x-3)^2} = 5(x-3)^{-2},$$

so

$$f'(x) = -\frac{10}{(x-3)^3}.$$

Thus the equation $f'(x) = 0$ has no solutions, so there are no stationary points.

4. The table of signs for $f'(x)$ is as follows.

x	$(-\infty, 3)$	3	$(3, \infty)$
-10	-	-	-
$(x-3)^3$	-	0	+
$f'(x)$	+	*	-

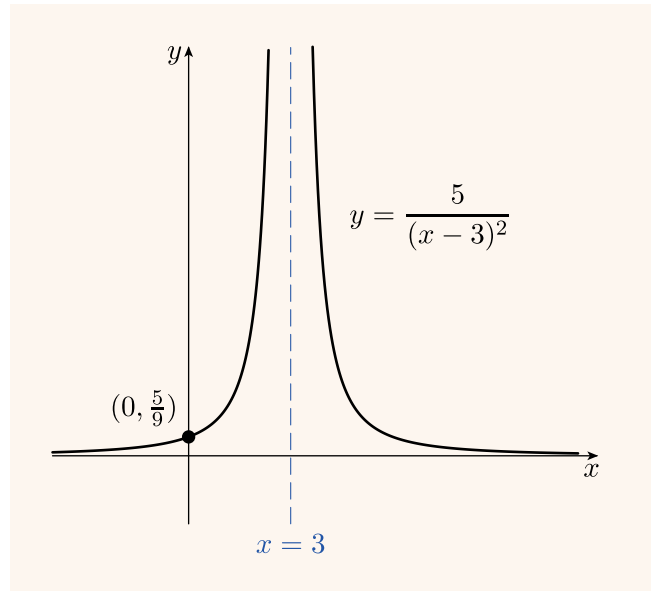
Hence f is increasing on the interval $(-\infty, 3)$ and decreasing on the interval $(3, \infty)$. That is, f is increasing on the left and decreasing on the right of the vertical asymptote $x = 3$.

5. Since $f(x)$ is a proper rational function, $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$, and the graph of f has a horizontal asymptote with equation $y = 0$.

Since f is increasing at the far left and decreasing at the far right, the graph of f approaches its horizontal asymptote from above at both the far left and the far right.

6. Since $f(1) = \frac{5}{4}$ and $f(-1) = \frac{5}{16}$, f is neither an even function nor an odd function. Note, however, that the graph of f is a horizontal translation by $+3$ of the function g with rule $g(x) = 5/x^2$, which is an even function. We can therefore conclude that the graph of f has the line $x = 3$ as an axis of symmetry.
7. The denominator of $f(x)$ is positive for all values of x in the domain of f , so f is always positive.

The graph of f is shown below.



- (b) The function is

$$f(x) = -\frac{x^3}{x^2 + 3}.$$

1. The domain of f is $\mathbb{R}$, and there are no vertical asymptotes.
2. The y -intercept is $f(0) = 0$.

Multiplying both sides of the equation $f(x) = 0$ by $-(x^2 + 3)$ gives

$$x^3 = 0, \quad \text{so} \quad x = 0.$$

Hence the only x -intercept is 0.

3. By the quotient rule,

$$\begin{aligned} f'(x) &= -\frac{(x^2 + 3) \times 3x^2 - x^3 \times 2x}{(x^2 + 3)^2} \\ &= -\frac{x^2(x^2 + 9)}{(x^2 + 3)^2}. \end{aligned}$$

Multiplying both sides of the equation $f'(x) = 0$ by $-(x^2 + 3)^2$ gives

$$x^2(x^2 + 9) = 0, \quad \text{so } x = 0.$$

Hence the only stationary point of f is 0. Since $f(0) = 0$, the coordinates of the stationary point are $(0, 0)$.

The table of signs for $f'(x)$ is as follows.

x	$(-\infty, 0)$	0	$(0, \infty)$
$-x^2$	—	0	—
$x^2 + 9$	+	+	+
$(x^2 + 3)^2$	+	+	+
$f'(x)$	—	0	—

Hence f is decreasing on the left and on the right of the stationary point, so the stationary point is a point of inflection.

4. (Not applicable.)
 5. We can divide the numerator of $f(x)$ by the denominator using polynomial long division, as follows.

$$\begin{array}{r} - x \\ x^2 + 0x + 3 \overline{) -x^3 + 0x^2 + 0x + 0} \\ \underline{-x^3 + 0x^2 - 3x} \\ 3x \end{array}$$

(Check:

$$\begin{aligned} -x(x^2 + 3) + 3x &= -x^3 - 3x + 3x \\ &= -x^3. \end{aligned}$$

Hence

$$f(x) = -x + \frac{3x}{x^2 + 3}.$$

It follows that $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$, and the graph of f has a slant asymptote with equation $y = -x$.

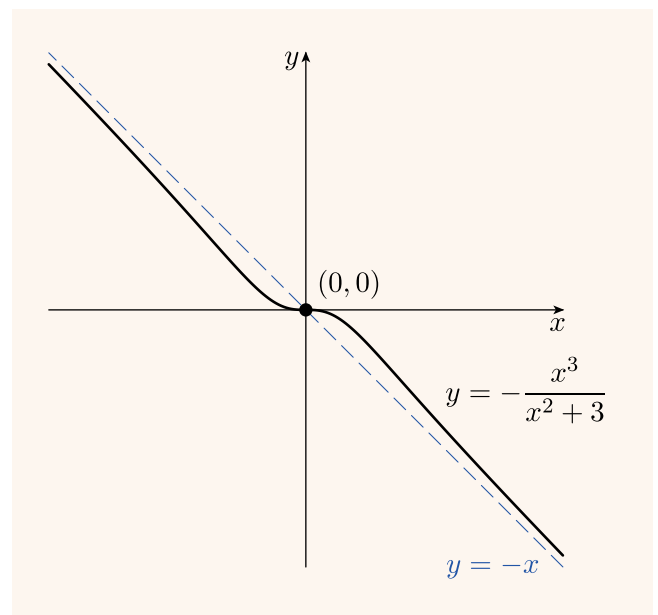
Since $3x/(x^2 + 3)$ is positive for $x > 0$ and negative for $x < 0$, the graph lies above this asymptote for $x > 0$ and below it for $x < 0$.

6. Since

$$\begin{aligned} f(-x) &= -\frac{(-x)^3}{(-x)^2 + 3} \\ &= -\left(-\frac{x^3}{x^2 + 3}\right) \\ &= -f(x), \end{aligned}$$

f is an odd function.

The graph of f is shown below.



- (c) The function is

$$f(x) = \frac{1}{x^2 + 4x + 3} = \frac{1}{(x + 3)(x + 1)}.$$

- The denominator $(x + 3)(x + 1)$ of $f(x)$ is 0 when $x = -3$ and $x = -1$. So the domain of f is $(-\infty, -3) \cup (-3, -1) \cup (-1, \infty)$, and the lines $x = -3$ and $x = -1$ are vertical asymptotes of the graph of f .
- The y -intercept is $f(0) = \frac{1}{3}$.

The equation $f(x) = 0$ is

$$\frac{1}{x^2 + 4x + 3} = 0,$$

which has no solutions. Hence there are no x -intercepts.

3. Writing $f(x) = (x^2 + 4x + 3)^{-1}$ and applying the chain rule gives

$$\begin{aligned} f'(x) &= -(x^2 + 4x + 3)^{-2}(2x + 4) \\ &= -\frac{2x + 4}{(x^2 + 4x + 3)^2} \\ &= -\frac{2x + 4}{((x + 3)(x + 1))^2}. \end{aligned}$$

Multiplying both sides of the equation $f'(x) = 0$ by $-((x + 3)(x + 1))^2$ gives

$$2x + 4 = 0, \quad \text{so} \quad x = -2.$$

Hence the only stationary point of f is -2 . Since $f(-2) = 1/(-1) = -1$, the coordinates of the stationary point are $(-2, -1)$.

For reasons of space the following table of signs for $f'(x)$ is split into two parts.

x	$(-\infty, -3)$	-3	$(-3, -2)$
$-(2x + 4)$	+	+	+
$(x + 3)^2$	+	0	+
$(x + 1)^2$	+	+	+
$f'(x)$	+	*	+

x	-2	$(-2, -1)$	-1	$(-1, \infty)$
$-(2x + 4)$	0	-	-	-
$(x + 3)^2$	+	+	+	+
$(x + 1)^2$	+	+	0	+
$f'(x)$	0	-	*	-

Hence f is increasing on the left and decreasing on the right of the stationary point, so the stationary point is a local maximum.

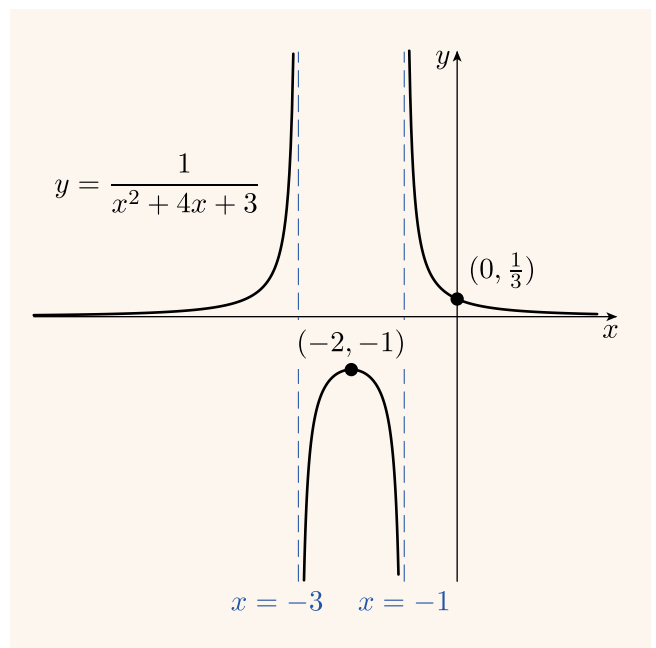
4. From the table of signs for $f'(x)$ we see that f is increasing on the intervals $(-\infty, -3)$ and $(-3, -2)$, and decreasing on the intervals $(-2, -1)$ and $(-1, \infty)$. That is, f is increasing on both sides of the vertical asymptote $x = -3$, and f is decreasing on both sides of the vertical asymptote $x = -1$.

5. Since $f(x)$ is a proper rational function, $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$, and the graph of f has a horizontal asymptote with equation $y = 0$.

Since f is increasing at the far left and decreasing at the far right, the graph of f approaches its horizontal asymptote from above at both the far left and the far right.

6. Since $f(2) = \frac{1}{15}$ and $f(-2) = -1$, f is neither an even function nor an odd function.

The graph of f is shown below.



- (d) The function is

$$f(x) = \frac{4x}{2x^2 + 3}.$$

- The domain of f is $\mathbb{R}$, and there are no vertical asymptotes.
- The y -intercept is $f(0) = 0$.

Multiplying both sides of the equation $f(x) = 0$ by $2x^2 + 3$ gives

$$4x = 0, \quad \text{so} \quad x = 0.$$

Hence the only x -intercept is 0.

3. By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(2x^2 + 3) \times 4 - 4x \times 4x}{(2x^2 + 3)^2} \\ &= \frac{12 - 8x^2}{(2x^2 + 3)^2}. \end{aligned}$$

Multiplying both sides of the equation $f'(x) = 0$ by $-(2x^2 + 3)^2$ gives

$$8x^2 - 12 = 0, \quad \text{so} \quad x = \pm \frac{\sqrt{6}}{2}.$$

Therefore the stationary points of f are $-\sqrt{6}/2$ and $\sqrt{6}/2$. Since

$$\begin{aligned} f\left(-\frac{\sqrt{6}}{2}\right) &= \frac{4(-\sqrt{6}/2)}{2(-\sqrt{6}/2)^2 + 3} \\ &= -\frac{2\sqrt{6}}{6} \\ &= -\frac{\sqrt{6}}{3} \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{\sqrt{6}}{2}\right) &= \frac{4(\sqrt{6}/2)}{2(\sqrt{6}/2)^2 + 3} \\ &= \frac{2\sqrt{6}}{6} \\ &= \frac{\sqrt{6}}{3}, \end{aligned}$$

the coordinates of the stationary points are $(-\sqrt{6}/2, -\sqrt{6}/3)$ and $(\sqrt{6}/2, \sqrt{6}/3)$.

The following table of signs for $f'(x)$ is split into two parts.

x	$\left(-\infty, -\frac{\sqrt{6}}{2}\right)$	$-\frac{\sqrt{6}}{2}$
$12 - 8x^2$	—	0
$(2x^2 + 3)^2$	+	+
$f'(x)$	—	0

x	$\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)$	$\frac{\sqrt{6}}{2}$	$\left(\frac{\sqrt{6}}{2}, \infty\right)$
$12 - 8x^2$	+	0	—
$(2x^2 + 3)^2$	+	+	+
$f'(x)$	+	0	—

Hence f is increasing on the interval $(-\sqrt{6}/2, \sqrt{6}/2)$ and decreasing on the intervals $(-\infty, -\sqrt{6}/2)$ and $(\sqrt{6}/2, \infty)$.

Since $f'(x)$ is negative to the left and positive to the right of the stationary point $-\sqrt{6}/2$, it follows that $-\sqrt{6}/2$ is a local minimum of f .

Since $f'(x)$ is positive to the left and negative to the right of the stationary point $\sqrt{6}/2$, it follows that $\sqrt{6}/2$ is a local maximum of f .

4. (Not applicable.)

5. Since $f(x)$ is a proper rational function, $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$, and the graph of f has a horizontal asymptote with equation $y = 0$.

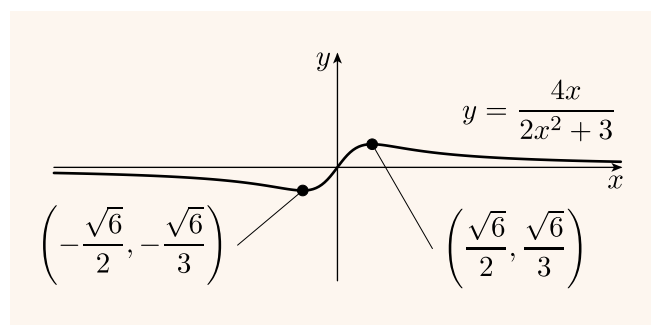
Since f is decreasing at both the far left and the far right, the graph of f approaches its horizontal asymptote from below at the far left and from above at the far right.

6. Since

$$\begin{aligned} f(-x) &= \frac{-4x}{2(-x)^2 + 3} \\ &= -\frac{4x}{2x^2 + 3} \\ &= -f(x), \end{aligned}$$

f is an odd function.

The graph of f is shown below.



Solution to Exercise 19

(a) By the product to sum identity for cosine,

$$\begin{aligned}
 & \int \cos(2x) \cos(3x) \, dx \\
 &= \int \left(\frac{1}{2} \cos(2x + 3x) + \frac{1}{2} \cos(2x - 3x) \right) dx \\
 &= \frac{1}{2} \int (\cos(5x) + \cos(-x)) \, dx \\
 &= \frac{1}{2} \int (\cos(5x) + \cos x) \, dx \\
 &= \frac{1}{2} \left(\frac{1}{5} \sin(5x) + \sin x \right) + c \\
 &= \frac{1}{10} (\sin(5x) + 5 \sin x) + c.
 \end{aligned}$$

(b) By the identity $\sin^2 x = 1 - \cos^2 x$,

$$\begin{aligned}
 & \int \sin^5 x \cos^4 x \, dx \\
 &= \int \sin^4 x \cos^4 x \sin x \, dx \\
 &= \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx.
 \end{aligned}$$

Let $u = \cos x$; then $\frac{du}{dx} = -\sin x$. The integral becomes

$$\begin{aligned}
 & - \int (1 - \cos^2 x)^2 \cos^4 x (-\sin x) \, dx \\
 &= - \int (1 - u^2)^2 u^4 \, du \\
 &= - \int (u^4 - 2u^6 + u^8) \, du \\
 &= - \left(\frac{1}{5} u^5 - \frac{2}{7} u^7 + \frac{1}{9} u^9 \right) + c \\
 &= -\frac{1}{5} \cos^5 x + \frac{2}{7} \cos^7 x - \frac{1}{9} \cos^9 x + c.
 \end{aligned}$$

(c) By the identity $1 + \cot^2 x = \operatorname{cosec}^2 x$,

$$\int \frac{\cot x}{1 + \cot^2 x} \, dx = \int \frac{\cot x}{\operatorname{cosec}^2 x} \, dx.$$

Let $u = \operatorname{cosec} x$; then

$$\frac{du}{dx} = -\operatorname{cosec} x \cot x.$$

The integral becomes

$$\begin{aligned}
 & - \int \frac{1}{\operatorname{cosec}^3 x} (-\operatorname{cosec} x \cot x) \, dx \\
 &= - \int \frac{1}{u^3} \, du \\
 &= \frac{1}{2u^2} + c \\
 &= \frac{1}{2 \operatorname{cosec}^2 x} + c \\
 &= \frac{1}{2} \sin^2 x + c.
 \end{aligned}$$

(The following alternative solution uses the double-angle identity for sine:

$$\begin{aligned}
 \int \frac{\cot x}{1 + \cot^2 x} \, dx &= \int \frac{\cot x}{\operatorname{cosec}^2 x} \, dx \\
 &= \int \frac{\cos x / \sin x}{1 / \sin^2 x} \, dx \\
 &= \int \sin x \cos x \, dx \\
 &= \int \frac{1}{2} \sin(2x) \, dx \\
 &= -\frac{1}{4} \cos(2x) + c.
 \end{aligned}$$

You may like to convince yourself that the results are equivalent!)

Solution to Exercise 20

(a) Let $x = 3 \sec u$; then $\frac{dx}{du} = 3 \sec u \tan u$.

Also, $u = \sec^{-1}(x/3)$. So

$$\begin{aligned}
 & \int \frac{\sqrt{x^2 - 9}}{2x} \, dx \\
 &= \int \frac{\sqrt{9 \sec^2 u - 9}}{6 \sec u} \times 3 \sec u \tan u \, du \\
 &= \int \frac{3 \sqrt{\sec^2 u - 1}}{2} \tan u \, du \\
 &= \frac{3}{2} \int \sqrt{\tan^2 u} \tan u \, du \\
 &= \frac{3}{2} \int \tan^2 u \, du \\
 &= \frac{3}{2} \int (\sec^2 u - 1) \, du \\
 &= \frac{3}{2} (\tan u - u) + c \\
 &= \frac{3}{2} \left(\tan \left(\sec^{-1} \left(\frac{x}{3} \right) \right) - \sec^{-1} \left(\frac{x}{3} \right) \right) + c.
 \end{aligned}$$

(b) Let $x = \tan u$; then $\frac{dx}{du} = \sec^2 u$. Also, $u = \tan^{-1} x$. So

$$\begin{aligned}
 & \int \frac{1}{x^2 \sqrt{x^2 + 1}} \, dx \\
 &= \int \frac{1}{\tan^2 u \sqrt{\tan^2 u + 1}} \sec^2 u \, du \\
 &= \int \frac{\sec^2 u}{\tan^2 u \sqrt{\sec^2 u}} \, du \\
 &= \int \frac{\sec u}{\tan^2 u} \, du \\
 &= \int \frac{\sec u}{\tan u} \times \frac{1}{\tan u} \, du \\
 &= \int \operatorname{cosec} u \cot u \, du \\
 &= -\operatorname{cosec} u + c \\
 &= -\operatorname{cosec}(\tan^{-1} x) + c.
 \end{aligned}$$

Solution to Exercise 21

(a) We have

$$\begin{aligned}x^2 - 4x + 5 &= (x - 2)^2 - 4 + 5 \\&= (x - 2)^2 + 1.\end{aligned}$$

Let $u = x - 2$; then $\frac{du}{dx} = 1$. So

$$\begin{aligned}\int \frac{1}{x^2 - 4x + 5} dx &= \int \frac{1}{(x - 2)^2 + 1} dx \\&= \int \frac{1}{u^2 + 1} du \\&= \tan^{-1} u + c \\&= \tan^{-1}(x - 2) + c.\end{aligned}$$

(b) The expression $2x - x^2$ in completed-square form is $1 - (x - 1)^2$.

Let $x - 1 = \sin u$, thus $x = \sin u + 1$.

Then $\frac{dx}{du} = \cos u$. Also, $u = \sin^{-1}(x - 1)$.

So

$$\begin{aligned}\int \frac{x + 3}{\sqrt{2x - x^2}} dx &= \int \frac{x + 3}{\sqrt{1 - (x - 1)^2}} dx \\&= \int \frac{\sin u + 4}{\sqrt{1 - \sin^2 u}} \cos u du \\&= \int \frac{\sin u + 4}{\sqrt{\cos^2 u}} \cos u du \\&= \int (\sin u + 4) du \\&= -\cos u + 4u + c \\&= -\cos(\sin^{-1}(x - 1)) \\&\quad + 4 \sin^{-1}(x - 1) + c.\end{aligned}$$

(c) We have

$$\begin{aligned}x^2 + 6x + 5 &= (x + 3)^2 - 9 + 5 \\&= (x + 3)^2 - 4.\end{aligned}$$

Let $x + 3 = 2 \sec u$, thus $x = 2 \sec u - 3$.

Then $\frac{dx}{du} = 2 \sec u \tan u$. Also,
 $u = \sec^{-1}((x + 3)/2)$. So

$$\begin{aligned}\int \frac{x + 3}{\sqrt{x^2 + 6x + 5}} dx &= \int \frac{x + 3}{\sqrt{(x + 3)^2 - 4}} dx \\&= \int \frac{2 \sec u}{\sqrt{4 \sec^2 u - 4}} \times 2 \sec u \tan u du \\&= \int \frac{4 \sec^2 u \tan u}{2 \sqrt{\sec^2 u - 1}} du \\&= 2 \int \frac{\sec^2 u \tan u}{\sqrt{\tan^2 u}} du \\&= 2 \int \sec^2 u du \\&= 2 \tan u + c \\&= 2 \tan \left(\sec^{-1} \left(\frac{x + 3}{2} \right) \right) + c.\end{aligned}$$

Solution to Exercise 22

$$\begin{aligned}\text{(a) } \sinh(\ln 2) &= \frac{1}{2}(e^{\ln 2} - e^{-\ln 2}) \\&= \frac{1}{2} \left(2 - \frac{1}{2} \right) \\&= \frac{3}{4}.\end{aligned}$$

$$\begin{aligned}\text{(b) } \tanh(\ln 3) &= \frac{e^{\ln 3} - e^{-\ln 3}}{e^{\ln 3} + e^{-\ln 3}} \\&= \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} \\&= \frac{\frac{8}{3}}{\frac{10}{3}} \\&= \frac{4}{5}.\end{aligned}$$

$$\text{(c) } \cosh(-5) = 74.2099 \text{ (to 4 d.p.)}.$$

$$\text{(d) } \sinh^{-1}(2) = 1.4436 \text{ (to 4 d.p.)}.$$

Solution to Exercise 23

$$\begin{aligned}\text{(a) } \sinh(2x) &= \frac{1}{2}(e^{2x} - e^{-2x}) \\&= \frac{1}{2}((e^x)^2 - (e^{-x})^2) \\&= \frac{1}{2}((e^x - e^{-x})(e^x + e^{-x})) \\&= 2 \left(\frac{1}{2}(e^x - e^{-x}) \times \frac{1}{2}(e^x + e^{-x}) \right) \\&= 2 \sinh x \cosh x.\end{aligned}$$

(b) $\coth^2 x - 1$

$$\begin{aligned}
 &= \frac{1}{\tanh^2 x} - 1 \\
 &= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^2 - 1 \\
 &= \frac{(e^x + e^{-x})^2}{(e^x - e^{-x})^2} - \frac{(e^x - e^{-x})^2}{(e^x - e^{-x})^2} \\
 &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x} - e^{2x} + 2e^x e^{-x} - e^{-2x}}{(e^x - e^{-x})^2} \\
 &= \frac{4e^0}{(e^x - e^{-x})^2} \\
 &= \frac{4}{(e^x - e^{-x})^2} \\
 &= \frac{1}{\left(\frac{1}{2}(e^x - e^{-x})\right)^2} \\
 &= \frac{1}{\sinh^2 x} \\
 &= \operatorname{cosech}^2 x.
 \end{aligned}$$

(c) We have

$$\frac{2 \tanh x}{1 + \tanh^2 x} = \frac{2 \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)}{1 + \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2}.$$

Multiplying the numerator and denominator by $(e^x + e^{-x})^2$ gives

$$\begin{aligned}
 &\frac{2(e^x - e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2 + (e^x - e^{-x})^2} \\
 &= \frac{2(e^{2x} + e^x e^{-x} - e^x e^{-x} - e^{-2x})}{e^{2x} + 2e^x e^{-x} + e^{-2x} + e^{2x} - 2e^x e^{-x} + e^{-2x}} \\
 &= \frac{2(e^{2x} - e^{-2x})}{2(e^{2x} + e^{-2x})} \\
 &= \tanh(2x).
 \end{aligned}$$

So

$$\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

Solution to Exercise 24

(a) By the chain rule,

$$f'(x) = 2 \operatorname{sech}^2(2x).$$

(b) By the product rule,

$$\begin{aligned}
 f'(x) &= \sinh x \operatorname{sech}^2 x + \tanh x \cosh x \\
 &= \sinh x (\operatorname{sech}^2 x + 1).
 \end{aligned}$$

(c) By the chain rule,

$$\begin{aligned}
 f'(x) &= e^{\cosh x} \times \sinh x \\
 &= \sinh x e^{\cosh x}.
 \end{aligned}$$

Solution to Exercise 25

(a) $\int \cosh(2x + 3) dx = \frac{1}{2} \sinh(2x + 3) + c.$

(b) $\int \cosh^3 x dx = \int \cosh^2 x \cosh x dx$
 $= \int (1 + \sinh^2 x) \cosh x dx.$

Let $u = \sinh x$; then $\frac{du}{dx} = \cosh x$. The integral becomes

$$\begin{aligned}
 \int (1 + u^2) du &= u + \frac{1}{3} u^3 + c \\
 &= \sinh x + \frac{1}{3} \sinh^3 x + c.
 \end{aligned}$$

(c) Integrating by parts gives

$$\begin{aligned}
 &\int x \sinh(2x) dx \\
 &= x \times \frac{1}{2} \cosh(2x) - \int 1 \times \frac{1}{2} \cosh(2x) dx \\
 &= \frac{1}{2} x \cosh(2x) - \frac{1}{4} \sinh(2x) + c.
 \end{aligned}$$

Solution to Exercise 26

(a) Let $x = 3 \sinh u$; then $\frac{dx}{du} = 3 \cosh u$.

Also, $u = \sinh^{-1}(x/3)$. So

$$\begin{aligned}
 &\int \frac{1}{\sqrt{9 + x^2}} dx \\
 &= \int \frac{1}{\sqrt{9 + 9 \sinh^2 u}} \times 3 \cosh u du \\
 &= \int \frac{3 \cosh u}{3 \sqrt{1 + \sinh^2 u}} du \\
 &= \int \frac{\cosh u}{\sqrt{\cosh^2 u}} du \\
 &= \int 1 du \\
 &= u + c \\
 &= \sinh^{-1} \left(\frac{x}{3} \right) + c.
 \end{aligned}$$

(b) We have

$$\sqrt{4x^2 - 1} = \sqrt{4\left(x^2 - \frac{1}{4}\right)} = 2\sqrt{x^2 - \frac{1}{4}}.$$

Let $x = \frac{1}{2} \cosh u$; then $\frac{dx}{du} = \frac{1}{2} \sinh u$.

Also, $u = \cosh^{-1}(2x)$. So

$$\begin{aligned} & \int \frac{1}{\sqrt{4x^2 - 1}} dx \\ &= \int \frac{1}{2\sqrt{x^2 - \frac{1}{4}}} dx \\ &= \int \frac{1}{2\sqrt{\frac{1}{4} \cosh^2 u - \frac{1}{4}}} \times \frac{1}{2} \sinh u du \\ &= \int \frac{\frac{1}{2} \sinh u}{2 \times \frac{1}{2} \sqrt{\cosh^2 u - 1}} du \\ &= \frac{1}{2} \int \frac{\sinh u}{\sqrt{\sinh^2 u}} du \\ &= \frac{1}{2} \int 1 du \\ &= \frac{1}{2} u + c \\ &= \frac{1}{2} \cosh^{-1}(2x) + c. \end{aligned}$$

(c) Let $x = \sinh u$; then $\frac{dx}{du} = \cosh u$. Also, $u = \sinh^{-1} x$. So

$$\begin{aligned} & \int \frac{x+1}{\sqrt{x^2+1}} dx \\ &= \int \frac{\sinh u + 1}{\sqrt{\sinh^2 u + 1}} \cosh u du \\ &= \int \frac{\sinh u + 1}{\sqrt{\cosh^2 u}} \cosh u du \\ &= \int (\sinh u + 1) du \\ &= \cosh u + u + c \\ &= \cosh(\sinh^{-1} x) + \sinh^{-1} x + c. \end{aligned}$$

Solution to Exercise 27

Each of the solutions to this exercise is preceded by a comment explaining how the method of integration was chosen.

(a) The integrand is a standard function of the linear expression $3x + 2$, so use the linear expression rule:

$$\begin{aligned} \int (3x + 2)^3 dx &= \frac{1}{3} \times \frac{1}{4} (3x + 2)^4 + c \\ &= \frac{1}{12} (3x + 2)^4 + c. \end{aligned}$$

(b) The integrand is again a standard function of the linear expression $3x + 2$:

$$\begin{aligned} \int \frac{1}{\sqrt{3x+2}} dx &= \int (3x+2)^{-1/2} dx \\ &= \frac{1}{3} \times 2(3x+2)^{1/2} + c \\ &= \frac{2}{3} \sqrt{3x+2} + c. \end{aligned}$$

(c) The derivative of $-x^2$ is $-2x$, which is the same as $4x$ apart from the coefficient, so the integrand can be written in the form

$$\begin{aligned} & \text{constant} \times e^{\text{something}} \\ & \times \text{the derivative of the something.} \end{aligned}$$

So use integration by substitution.

Let $u = -x^2$; then $\frac{du}{dx} = -2x$. So

$$\begin{aligned} \int 4xe^{-x^2} dx &= -2 \int e^{-x^2} (-2x) dx \\ &= -2 \int e^u du \\ &= -2e^u + c \\ &= -2e^{-x^2} + c. \end{aligned}$$

(d) The integrand involves the square root of an expression of the form $a^2 + x^2$, so try a hyperbolic substitution.

Let $x = 3 \sinh u$; then $\frac{dx}{du} = 3 \cosh u$.

Also, $u = \sinh^{-1}(x/3)$. So

$$\begin{aligned} & \int \sqrt{9 + x^2} dx \\ &= \int \sqrt{9 + 9 \sinh^2 u} (3 \cosh u) du \\ &= \int 3\sqrt{1 + \sinh^2 u} (3 \cosh u) du \\ &= 9 \int \sqrt{\cosh^2 u} (\cosh u) du \\ &= 9 \int \cosh^2 u du. \end{aligned}$$

By the half-variable identity for cosh, the integral becomes

$$\begin{aligned} & 9 \int \frac{1}{2} (\cosh(2u) + 1) du \\ &= \frac{9}{2} \left(\frac{1}{2} \sinh(2u) + u \right) + c \\ &= \frac{9}{4} \sinh \left(2 \sinh^{-1} \left(\frac{x}{3} \right) \right) + \frac{9}{2} \sinh^{-1} \left(\frac{x}{3} \right) + c. \end{aligned}$$

- (e) The integrand involves an expression of the form $a^2 - x^2$, so try a trigonometric substitution.

Let $x = 2 \sin u$; then $\frac{dx}{du} = 2 \cos u$. Also, $u = \sin^{-1}(x/2)$. So

$$\begin{aligned} & \int \frac{x^2}{\sqrt{4-x^2}} dx \\ &= \int \frac{4 \sin^2 u}{\sqrt{4-4 \sin^2 u}} (2 \cos u) du \\ &= \int \frac{8 \sin^2 u \cos u}{2 \sqrt{1-\sin^2 u}} du \\ &= 4 \int \frac{\sin^2 u \cos u}{\sqrt{\cos^2 u}} du \\ &= 4 \int \sin^2 u du. \end{aligned}$$

By the half-angle identity for sine, the integral becomes

$$\begin{aligned} & 4 \int \frac{1}{2} (1 - \cos(2u)) du \\ &= 2 \left(u - \frac{1}{2} \sin(2u) \right) + c \\ &= 2 \sin^{-1} \left(\frac{x}{2} \right) - \sin \left(2 \sin^{-1} \left(\frac{x}{2} \right) \right) + c. \end{aligned}$$

- (f) The integrand is a rational expression, so find its partial fraction expansion.

We have

$$\begin{aligned} \frac{1}{x^2 + 7x + 12} &= \frac{1}{(x+3)(x+4)} \\ &= \frac{A}{x+3} + \frac{B}{x+4}, \end{aligned}$$

where A and B are constants.

Using the cover-up method with $x = -3$ gives

$$A = \frac{1}{-3+4} = 1.$$

Using the cover-up method with $x = -4$ gives

$$B = \frac{1}{-4+3} = -1.$$

Therefore

$$\frac{1}{x^2 + 7x + 12} = \frac{1}{x+3} - \frac{1}{x+4}.$$

(Check: when $x = 0$, LHS = $\frac{1}{12}$ and RHS = $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.)

So

$$\begin{aligned} & \int \frac{1}{x^2 + 7x + 12} dx \\ &= \int \left(\frac{1}{x+3} - \frac{1}{x+4} \right) dx \\ &= \ln |x+3| - \ln |x+4| + c \\ &= \ln \left| \frac{x+3}{x+4} \right| + c. \end{aligned}$$

- (g) The integrand can be modified so that it involves an expression of the form $x^2 + a^2$, so try a trigonometric substitution.

We have

$$\int \frac{1}{2x^2 + 6} dx = \frac{1}{2} \int \frac{1}{x^2 + 3} dx.$$

Let $x = \sqrt{3} \tan u$; then $\frac{dx}{du} = \sqrt{3} \sec^2 u$.

Also, $u = \tan^{-1}(x/\sqrt{3})$. So the integral becomes

$$\begin{aligned} & \frac{1}{2} \int \frac{1}{3 \tan^2 u + 3} (\sqrt{3} \sec^2 u) du \\ &= \frac{1}{2} \int \frac{\sqrt{3} \sec^2 u}{3(\tan^2 u + 1)} du \\ &= \frac{\sqrt{3}}{6} \int \frac{\sec^2 u}{\sec^2 u} du \\ &= \frac{\sqrt{3}}{6} \int 1 du \\ &= \frac{\sqrt{3}}{6} u + c \\ &= \frac{\sqrt{3}}{6} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + c. \end{aligned}$$

- (h) The integrand is an improper rational expression, so find its partial fraction expansion.

$$\begin{array}{r} x-2 \\ x^2+2x+1 \overline{) x^3+0x^2+0x+4} \\ \underline{x^3+2x^2+x} \\ -2x^2-x+4 \\ \underline{-2x^2-4x+2} \\ 3x+6 \end{array}$$

So the quotient on dividing $x^3 + 4$ by $x^2 + 2x + 1$ is $x - 2$, and the remainder is $3x + 6$. Hence

$$\begin{aligned}\frac{x^3 + 4}{x^2 + 2x + 1} &= x - 2 + \frac{3x + 6}{x^2 + 2x + 1} \\ &= x - 2 + \frac{3x + 6}{(x + 1)^2}.\end{aligned}$$

Now

$$\frac{3x + 6}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2},$$

where A and B are constants.

Using the cover-up method with $x = -1$ gives

$$B = \frac{3 \times (-1) + 6}{1} = 3.$$

So

$$\frac{3x + 6}{(x + 1)^2} = \frac{A}{x + 1} + \frac{3}{(x + 1)^2}.$$

Multiplying both sides by $(x + 1)^2$ gives

$$3x + 6 = A(x + 1) + 3.$$

Substituting $x = 0$ gives

$$6 = A + 3, \quad \text{so } A = 3.$$

Hence

$$\frac{3x + 6}{(x + 1)^2} = \frac{3}{x + 1} + \frac{3}{(x + 1)^2}.$$

So

$$\frac{x^3 + 4}{x^2 + 2x + 1} = x - 2 + \frac{3}{x + 1} + \frac{3}{(x + 1)^2}.$$

(Check: when $x = 0$, LHS = 4 and RHS = $-2 + 3 + 3 = 4$.)

Therefore

$$\begin{aligned}\int \frac{x^3 + 4}{x^2 + 2x + 1} dx &= \int \left(x - 2 + \frac{3}{x + 1} + \frac{3}{(x + 1)^2} \right) dx \\ &= \frac{1}{2}x^2 - 2x + 3 \ln|x + 1| - \frac{3}{x + 1} + c.\end{aligned}$$

- (i) The expression $x - 4$ becomes simpler when differentiated, and the expression $\sin(3x)$ can be integrated. So try integration by parts:

$$\begin{aligned}\int (x - 4) \sin(3x) dx &= (x - 4) \times \frac{1}{3}(-\cos(3x)) \\ &\quad - \int 1 \times \frac{1}{3}(-\cos(3x)) dx \\ &= -\frac{1}{3}(x - 4) \cos(3x) + \frac{1}{3} \int \cos(3x) dx \\ &= \frac{1}{3}(4 - x) \cos(3x) + \frac{1}{3} \times \frac{1}{3} \sin(3x) + c \\ &= \frac{1}{3}(4 - x) \cos(3x) + \frac{1}{9} \sin(3x) + c.\end{aligned}$$

- (j) The identity $\sin 2\theta = 2 \sin \theta \cos \theta$ can be used to rewrite the integrand:

$$\begin{aligned}\int 4 \sin(2x) \cos(2x) dx &= 2 \int \sin(4x) dx \\ &= 2 \times \left(-\frac{1}{4} \cos(4x) \right) + c \\ &= -\frac{1}{2} \cos(4x) + c.\end{aligned}$$

- (k) The derivative of $4x^2 + 1$ is $8x$, which is the same as $3x$ apart from the coefficient, so the integrand can be written in the form

$$\text{constant} \times \frac{1}{\text{something}} \times \text{the derivative of the something}.$$

So use integration by substitution.

Let $u = 4x^2 + 1$; then $\frac{du}{dx} = 8x$. So

$$\begin{aligned}\int \frac{3x}{4x^2 + 1} dx &= \frac{3}{8} \int \frac{8x}{4x^2 + 1} dx \\ &= \frac{3}{8} \int \frac{1}{u} du \\ &= \frac{3}{8} \ln|u| + c \\ &= \frac{3}{8} \ln(4x^2 + 1) + c.\end{aligned}$$